Learning mathematics as developing a discourse[1]

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1. Posing the question: What is it that changes when one learns mathematics?

In the field of mathematics education, the terms *discourse* and *communication* seem to be these days in everybody's mouth. They feature prominently in research papers, they can be heard in teacher preparation courses, and they appear time and again in variety of programmatic documents that purport to establish instructional policies (see e.g. *Principles and standards for school mathematics*, NCTM 2000). All this could be interpreted as showing merely that we became as aware as ever of the importance of mathematical conversation for the success of mathematical learning. In this talk, I will try to show that there is more to discourse than meets the ears, and that putting communication in the heart of mathematics education is likely to change not only the way we teach but also the way we think about learning and about what is being learned. Above all, I will be arguing that communication should be viewed not as a mere aid to thinking, but as almost tantamount to the thinking itself. The *communicational approach to cognition*, which is under scrutiny in this talk, is built around this basic theoretical principle.

To begin with, let us pay a brief visit to two classrooms where learning of a new mathematical topic has just started. The first class is just being introduced to the concept of negative number. The teacher takes her place in the front of the group of twelve-year old seven graders and initiates the conversation[2].

Episode 1: The first lesson on negative numbers

[N1] Teacher:	Have you ever heard about negative numbers? Like in temperatures, for example?
[N2] Omri:	Minus!
[N3] Teacher:	What is minus?
[N4] Roy:	Below zero.
[N5] Teacher:	*Temperature* below zero?
[N6] Sophie:	Below zero… it can be minus five, minus seven… Any number.
[N7] Teacher:	Where else have you seen positive and negative numbers?
[N8] Omri:	In the bank.
[N9] Teacher:	And do you remember the subject "Altitude"? What is *sea level*?
[N10] Yaron:	Zero
[N11] Teacher:	And above sea level? More than zero?
[N12] Yaron:	From one meter up.

Since we are interested in learning, and learning means change, we may analyze this episode by trying to describe the modifications that have yet to occur in the children's ways of dealing with the negative numbers. At the first sight, this future learning is not just a matter of a change; rather, it requires creating something completely new. The children, although not entirely ignorant of negative numbers, can

[1] Sfard, A. (2001). Learning mathematics as developing a discourse. In R. Speiser, C. Maher, C. Walter (Eds), *Proceedings of 21st Conference of PME-NA* (pp. 23-44). Columbus, Ohio: Clearing House for science, mathematics, and Environmental Education.

[2] These data are taken from the study conducted with Sharon Avgil. This and all the subsequent segments of transcripts have been translated from Hebrew by the author.

do little more at the moment than associate the topic with certain characteristic terms, such as *minus* or *below zero*. It seems, therefore, that they will have to work on the subject almost from scratch. To put it in the traditional language, we may say that the children are yet to *acquire the concept of negative number* or *to construct this concept* for themselves.

Rather than trying to fathom the operative meaning of these last words, let me now turn to another episode, in which two first graders, Shira and Eynat, begin learning some basic geometry. The girls are first shown a number of geometric figures and are asked by the teacher to mark those that can be called triangles. Once the task is completed, the following conversation between the girls and the teacher takes place[3]:

Episode 2: The first meeting about triangles

[T1] Eynat:	[*Pointing to shape A*] This is a triangle but is also has other lines.
[T2] Teacher	Well, Eynat, how do you know that triangle is indeed a triangle?
[T3] Eynat:	Because it has three..aah…three… well.. lines.
…….	…….
[T22] Teacher:	[*Pointing to shape B*] This one also: one, two three..
[T23] The girls:	Yes
[T24] Teacher:	So, it *is* triangle? Why didn't you mark it in the beginning?
[T25] Eynat:	'Cause then… I did not exactly see it.. I wasn't sure [*While saying this, Eynat starts putting a circle also around shape C*]
…….	……….
[T28] Shira:	[*Looking at shape C that Eynat is marking*] Hey, this is not a triangle. Triangle is wide and this one is thin.
[T29] Eynat:	So what? [*but while saying this, she stops drawing the circle*]
[T30] Teacher:	Why? Why this is not a triangle [points to shape B]? Shira said it is too thin. But haven't we said…
[T31] Eynat:	There is no such thing as too thin. [*but while saying this, she erases the circle around shape C*]
[T32] Teacher:	Triangle -- must it be of a certain size?
[T33] Shira:	Hmmmm…Yes, a little bit… It must be wide. What's that? This is not like a triangle – this is a stick!

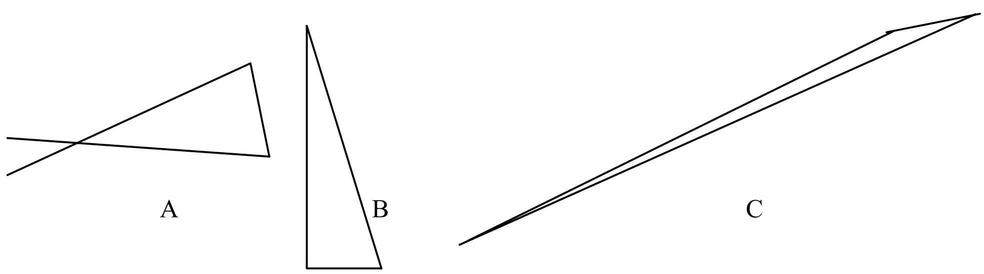

Fig. 1: Triangle or not triangle?

Here, unlike in the case of negative numbers, the students are already well acquainted with the mathematical objects in question, the triangles. And yet, neither the way they speak about these shapes nor the manner in which they act with them is fully satisfactory from the point of view of the teacher. In her

[3] These data are taken from a study conducted with Orit Shalit-Admoni and Pnina Shavit.

search for triangles Shira disqualifies any shape that seems to her too thin. Eynat, even though aware of the formal definition of triangles (T3) and apparently convinced that "there is no such thing as too thin" (T31), still cannot decide whether the stick-like shape in the picture is a triangle or not: Originally, she has not marked shape C as triangle; now, following the conversation on shape B, she encircles this shape (T25), only to change her mind again (T29), and to erase the circle, eventually (T31). The teacher will be eager to induce some changes in the ways children think, speak and act with triangles. The development she is anxious to see is of a different kind than the one required in the case of negative numbers. Still, we can describe this new change in terms of concept acquisition and conceptual change, just as we did before: We can say that the children face the formidable task of *overcoming their misconceptions about triangles*.[4]

In this talk, I will reformulate this last statement after introducing a somewhat different way of talking about learning. My preference for the framework that will be called *communicational* stems mainly from the conviction that theories which conceptualize learning as personal acquisitions can tell us only so much about the complex phenomenon of learning. The acquisitionist approach relies heavily on the idea of cognitive invariants that cross cultural and situational borders. Consequently, the theories that come from this tradition are geared toward finding and investigating what remains constant when the situation changes. And yet, as has been convincingly argued by many scholars (e.g. Lave, 1988; Cole, 1996), human learning is too dynamic and too sensitive to ongoing social interactions to be fully captured in the terms of decontextualized mental schemes, built according to universal rules. In fact, my point of departure in this talk is that most of our *learning is nothing else than a special kind of social interaction aimed at modification of other social interactions*. Thus, rather than looking for those properties of the individual that can be held responsible for the constancy of this person's behavior, I am opting for a framework that allows me to stay tuned to the interactions from which the change arises. Let me add however, that my choice of the framework should not be interpreted as a rejection of the long-standing acquisition metaphor. The communicational approach should be regarded as a framework that has a potential to subsume this more traditional outlook, while modifying its hidden epistemological infrastructure.

2. Communicational approach to learning

Let me go back to the two episodes we have just seen and try to describe the required change without having recourse to invisible mental structures stored in the students' heads[5]. While listening to the two brief conversations between the children and their teachers we had good reasons to wonder about the quality of the communication that was taking place. In the first scene, although it was obvious that the children were already familiar with the key term *negative number*, it was also clear that they could not say much about the topic of the exchange. It is significant that they answered teacher's questions with single- or double-word exclamations, such as "Minus!" or "Below zero", rather than in full sentences. We can say that at this point, the students could identify the discourse on negative numbers when they heard it, but they were not yet able to take an active part in it. In the second episode the situation, although different, still asked for a change. True, the children eagerly participated in the discourse on triangles; and yet, the way they did this was unlike that of their teacher.

It is important to note that while introducing children to new ways of communicating seems to be the teacher's principal goal, the work never starts from zero. Whether the discourse to be taught is on negative numbers or on triangles, it will be developed out of the discourses in which the children are already fluent. Thus, whatever the topic of learning, the teacher's task is to modify and exchange the existing discourse rather than to create a new one form scratch. If so, we can define learning as the *process of changing one's discursive ways in a certain well-defined manner*. More specifically, a person who learns about triangles or negative numbers alters and extends her discursive skills so as to become able to communicate on these topics with members of mathematical community. The new discourse may also be expected to make it possible to solve problems that could not be solved in the past.

At this point somebody may object and say that there is more to learning than modifying communication. Learning, the critic would say, is first and foremost about changing the ways we *think*, and

[4] Alternatively, in this latter case we may say, inspired by Vygotsky (1987), that the teacher tries to help children in making the transition from *spontaneous* to *scientific* concept of triangle.

[5] Let me stress once again: a statement like this should not be read as a denial of the existence of mental structures. Rather, it is a methodological claim; it is a declaration that, as a researcher, I feel on a firmer ground when I can base my arguments on observable aspects of the phenomena under inquiry.

the issue of how we communicate this thinking, although important, is still of only secondary significance. Let me then argue that thinking has not been excluded from my communicational account of learning. This point becomes immediately clear when we realize that the traditional split between thinking and communicating is untenable, and that *thinking is a special case of the activity of communicating.*[6] Indeed, a person who thinks can be seen as communicating with herself. This is true whether the thinking is in words, in images, or in any other symbols. Our thinking is clearly a dialogical endeavor, where we inform ourselves, we argue, we ask questions, and we wait for our own response. If so, becoming a participant in mathematical discourse is tantamount to learning to *think* in a mathematical way.

Let us now go back to our two classroom episodes and reformulate the initial query in communicational terms. Asking what the children have yet to learn is now equivalent to inquiring how students' way of communicating should change if they are to become skilful participants of mathematical discourse on triangles and negative numbers. Let me begin with the younger students. Clearly, Eynat and Shira have to modify their use of the keyword *triangle*. This seemingly superficial and rather marginal change is, in fact, quite profound and by no means easy to implement. Indeed, this change will not occur unless the students adopt new criteria for judging the appropriateness of the word use. So far, the children's decisions to call different shapes the same name *triangle* was based mainly (at least by Shira) or also (as in the case of Eynat) on the their perception of the overall visual similarity of these shapes. This is true whether the shapes were actually seen or just remembered. If Shira had difficulty identifying the thin shape in Fig. 1 as a triangle, it is because it struck her with its similarity to a stick. According to the rules that govern the child's discourse at the moment, different names mean different shapes, and thus what is called stick cannot be called triangle as well. The mapping between things and names must be a single-valued function. This is the only possibility in the world where names are part and parcel of the shapes and like the shapes themselves, are externally given rather than being products of human decisions. All this will have to change in the process of learning. From now on, the students will have to seek the advice of verbal definitions and treat the latter as the exclusive basis for their decisions about what can count as "the same" or as different. These decisions will be mediated by language and will involve certain well-defined procedures. Among others, before the children decide about the name that should be given to a shape, they will have to scan this shape in a linear way, splitting it in separate parts and counting the thus obtained elements (note that counting is a verbal action without which the new kind of decision procedure, and thus the new type of sameness, would not be possible). This is a far- reaching change, one that affects the *meta-discursive rules* that regulate the ways discursive decisions are made.[7]

In the case of negative numbers, even more extensive changes are required. When it comes to words, it is not just a matter of modifying their use. The students will have to extend their vocabulary and to learn to operate with such new terms as "negative two" or "negative three and a half". Unlike in the case of triangles, where one can identify the object of talk with the help of pictures (e.g. triangles drawn on the paper), the students will now need new, specially designed visual means to mediate the communication. Some special symbols, such as -2 or -3.5, and geometric models, such as extended number line, will soon be introduced. Like in the case of triangles, a change will also be required at the meta-discursive level. I will elaborate on this last theme later.

For now, let me generalize these last observations. The analysis of the two episodes has shown that the children's present discourse differs from typical school discourse along at least three dimensions:

- its *vocabulary*
- the visual means with which the communication is mediated, and

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[6] The communicational approach presented in this talk is similar to, although not identical with, the *discursive psychology* promoted, among others, by Harre & Gillett (1994) and by Edwards (1997).

[7] Note that my present description of the required change is quite similar – one can say isomorphic – to the one that could be given based on the van Hiele theory of the development of geometrical thinking (van Hiele 1985). Still, the two descriptions are put apart by their different epistemological/ontological underpinnings: While van Hiele's analysis, firmly rooted in the Piagetean framework, would produce a story of mental schemes, the present description is the description of students' ways of communicating. What makes the latter version qualitatively different from the former one is that it presents the development of child's geometrical thinking as a part and parcel of the development of her communicational skills, and thus makes salient the principal role of language, of contextual factors and of social interaction.

- the *meta-discursive rules* that navigate the flow of communication and tacitly tell the participants what kind of discursive moves would count as suitable for this particular discourse, and which would be deemed inappropriate.

Thus, if learning mathematics is conceptualized as a development of a mathematical discourse, to investigate learning means getting to know the ways in which children modify their discursive actions in these three respects. In the rest of this talk I will be analyzing the ways in which the required change can take place. While doing this I hope to show that adoption of the communicational approach to cognition is not an idle intellectual game and that it influences both our understanding of what happens when children learn mathematics and our ideas about what should be done to help students in this endeavor.

3 How do we create new uses of words and mediators?

According to a popular, commonsensical vision of the sequence of events that take place in the course of learning, the student must first have an idea of a new mathematical object, then give this idea a name and, eventually, he or she must also practice its use. This picture of learning may well be one that underlies the principle of "learning with understanding" that stresses the importance and primacy of conceptual understanding over formalization and skill (see e.g. Hiebert & Carpenter, 1992). The child is supposed to understand a mathematical idea, at least to some extent, before she starts using special mathematical names and symbols that "represent" it, and before she becomes proficient in these uses.

Conceptualization of learning as an introduction to a discourse leads me to doubt this popular model and makes the case for a different course of learning. Let us take the learning of negative numbers as an example. I will be arguing now that introduction of new names and new signifiers is the beginning rather than the end of the story. First, let me show the virtual impossibility of teaching a new discourse without actually speaking about its objects from the very first moment. Let us have a look at the way in which negative numbers are introduced in a school textbook (see Fig. 2).

> Let's choose a point on a straight line and name it "zero." Let's choose a segment and call it "the unit of length." Let's place the unit head-to-tail repeatedly on the line to the right of the point "zero." *The points made this way will be denoted by 1, 2, 3 and so on …*
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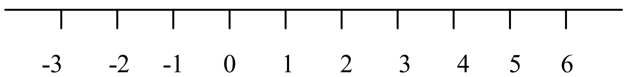
> 
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> To the left of the point "zero," we put the unit segment head-to-tail again and *denote the points obtained in this way with numbers -1, -2, -3,... The set of numbers created in this way is called the set of negative numbers*.

Fig. 2: From a school textbook (Mashler, M., 1976, *Algebra for 7th grade*). Translated from Hebrew.

The crux of this definition is in the interesting conceptual twist: points on the number axis are marked with decimal numerals preceded by dash and, subsequently, they are called *negative numbers.* One may wonder how this verbal acrobatics – giving *new names* to *points* and saying these are *numbers* – can enable the child an access to a discourse on the negatives. At the first sight, the learning sequence that begins with giving a new name to an old thing seems somehow implausible. And yet, such order of things may be inevitable, and it may also be more effective than we tend to think.

It is *inevitable* because in order to initiate children to a discourse on new objects, one already has to use this discourse. The objects of the discourse must thus be identified in one way or another, in words or symbols. This is probably why the teacher in *Episode 1* cannot refrain from using words like "negative numbers", "minus two", etc. while introducing the topic for the first time. Clearly, she feels compelled to do it in spite of the fact that the children have little idea abut the uses to which these words can be put. The proposed order of things in the process of learning is also more *effective* than we tend to think because of the simple fact that the new objects – the negatives – have been associated, and introduced with, the word *number.* The familiar notion evokes in the student expectations with respect to the possible uses of the new signifiers, such as -1, or -2.5. The children know that some numerical operations are involved. They know that many rules that hold for numbers will now hold for the negatives. For better for worse, the children seem to know quite a lot about this something to which they might have been exposed through a single sentence. In the episodes that we are going to see next we will have a chance to see how they make use of their former discursive experience with numbers.

Let us return to our seven graders then, to see how well they are doing as the newcomers to the discourse on negative numbers. In the new classroom scene that follows, we can see how the expectations evoked by he word *number* help the students find their ways into the new discourse. Some of these ways are like those of the expert participants, and some have to be deemed mistaken. At the present stage, three weeks and sixteen one-hour meetings later, the children already know how to add signed numbers and are trying to figure out for themselves how to multiply a positive by negative. First, they do it in small groups. In one of these groups, the following exchange takes place after the teacher asked what $2\cdot(-5)$ could be equal to:

Episode 3: The teacher asked what $2\cdot(-5)$ could be equal to.

[N13] Sophie:	Positive two times negative five…
[N14] Ada:	Two times negative five..
[N15] Sophie:	Aha, hold on… hold on… It's as if you said negative five multiplied two times…. So, negative five multiplied two times it's negative ten…

So far, so good. By projecting in metaphorical manner from their former discursive experience, the children discovered for themselves the rule which is, indeed, generally accepted. I will now show that this is not always the case. During the classroom discussion that took place after the work in pairs was completed, the following exchange took place in response to the same question as before:

Episode 4: In response to the question, "What $2\cdot(-5)$ could be equal to?"

[N16] Roy:	Negative ten.
[N17] Teacher:	Why?
[N18] Roy:	We simply did… two times negative five equals negative ten because five is the bigger number, and thus… uhmm… It's like two times five is ten, but [it's] negative ten because it is negative five.
……	
[N42] Noah:	And if it was the positive seven instead of positive two?
[N43] Yoash:	Then there will be positive thirty five
[N44] Sophie:	Why?
[N45] Yoash:	Because the plus [the positive] is bigger.

On the first sight, Roy's idea may sound somehow surprising. On the closer look, it is as justified as the one proposed by Sophie: like the girl before him, Roy draws on previously developed discursive habits, except that this time the choice does not fit with the one made along history by the mathematical community. Indeed, in the first case, the children substitute the new numbers for old numbers: The negatives slid into the slot of the second multiplier, occupied so far exclusively by unsigned numbers. In the second case, the students substituted operation for operation: The *multiplication* of signed numbers was obtained from the multiplication of unsigned numbers more or less in the way in which the *addition* of signed numbers has been previously obtained from the addition of the unsigned (see a symbolic presentation of the templates in Fig. 3). As already noted, while the choice of the first group may be deemed successful because it happens to adhere to what counts as proper in the mathematical discourse, the choice of the other group fails to meet the standards. What is most important, however, from the researcher's point of view, is the similarity between the two cases rather than the difference: in both episodes we have seen students trying to incorporate the newly encountered negatives into the discourse on numbers, and in both episodes they did it by using old discursive templates for the new signifiers.

Successful try: substitution into the discursive template

$$a \cdot b = b + b$$

Unsuccessful try: substitution into the discursive template

$$(+a) + (-b) = \begin{cases} |a - b| & \text{if } a > b \\ -\ |a - b| & \text{if } a \leq b \end{cases}$$

in which *a* and *b* are "unsigned" and both + and – are substituted with $\cdot$

Fig. 3: Recycling old discursive templates in the new context

Let me now go beyond the present examples and speculate on two distinct stages of learning likely to follow the introduction of a new signifier. As we have just seen, the use of a new signifier is at first *template-driven*, that is, based on substituting new signifiers into old discursive templates. The mechanism of recycling old discursive habits, bearing a family resemblance to Lakoff's conceptual metaphor (Lakoff 1993), is at work here. This stage is characterized by a rather inflexible use of the new signifier and by treating symbols, such as -5, as things in themselves, not standing for anything else. If all goes well, and if there is no visible to reason to renounce the templates that have been chosen, both these features will eventually disappear thanks to what will be called here *objectification* of the discourse. The signifiers will eventually begin to be referred to as *representations* of another entities believed to have independent existence of sorts. This is when, for the student, -5 becomes but a representation of a *negative number*, with the latter conceived as intangible object that can also be represented in many other ways (e.g. as 15–20 or as a point on the number axis) and which exists independently of the human mind. At this stage, the use of the new signifiers becomes much more flexible. New number words and symbols get life of their own and they can now be incorporated into linguistic structures that were unheard of within the old, more restricted discourse. The new discourse makes it also possible to say much more in a much smaller number of words.

This two-phase process of discourse development, summarized in Fig. 4, is kept in motion by what I once described metaphorically as a mechanism of a pump (Sfard, 2000a): Introduction of a symbol is like lifting the piston in that it creates a new semantic space – a need for a new meaning, a new discursive habit. The gradual objectification of the discourse is analogous to the procedure of filling in the space thus created. It is through an intermittent creation of a space "hungry" for new objects and through its subsequent replenishment with new discursive forms and relations that the participants of mathematical discourse steadily expand its limits.

Stage one: Template-driven use of signifiers
Templates *come in clusters*
The new signifier -- *only within certain phrases.*
Only a *weak sense* of signified (object)
Stage two: Objectified use of symbols
A symbol becomes a representation
Other signifiers are used as equivalent
Flexibility and generality
Economy of expression
Experience (sense of) a signified (object)

Fig.4. "Pump mechanism": Two phases in extending a mathematical discourse (adding new objects)

Turning to old discursive habits may be the only way to deal with the somewhat paradoxical nature of mathematical learning. At a closer look, the process of objectification turns to be inherently circular: If mathematical objects, such as negative numbers, are discursive constructions, we have to talk about them in order to bring them into being. On the other hand, how can we talk about something that does not yet exist for us? Or, to put it differently, signifieds can only be built through discursive use of the signifiers, but at the same time, the existence of these signifieds is a prerequisite for the meaningful use of the signifiers. This circularity, although an infallible source of difficulty and a serious trap for the newcomers to a discourse, is in fact the driving force behind this discourse's incessant growth. This is what fuels the process of co-emergence in which the new discursive practices and the new signifieds spur each other's development. In this process, the discursive forms and meanings, as practiced and experienced by interlocutors, are like two legs which make moving forward possible due to the fact that at any given time one of them is ahead of the other.

To sum up, in the Episodes 3 and 4 we have seen students who, in their attempt to develop the discourse on negative numbers, have just entered the template-driven phase. The dilemma which of the two possible templates for multiplying positive number by negative has not yet been resolved. The teacher temporarily refrains from giving any advice and the children, left to themselves, have yet a long way to go. The situation with the girls who are learning the about triangles is quite different, and it is different in two crucial respects. First, the children's discourse on geometrical shapes, although quite unlike that of their teacher, is already objetified. Indeed, for the girls triangles are externally given objects and not an arbitrary construct the identity of which can be discursively decided. Second, quite unlike in the former example, it is now the teacher who will try to establish new uses of the word triangle and will attempt to change the rules that govern these uses. Thus, in this case, the teacher is an active initiator of new discursive habits.

4. How do we create new meta-discursive rules and turn them our own?

So far, we have been focusing on discursive changes that take place following an extension of vocabulary (e.g. introduction of number names, such as "negative one" or "negative ten"), an addition of new mediating means (such as new numerical symbols or extended number line) or an alteration of word use (Shira's teacher proposes to apply the word 'triangle' to a shape the girls are reluctant to call this name). We have been talking about two developmental phases in the use of new discursive means: the phase of template-driven use and the phase of objectified use. It is now important to recall that together with the alterations in use, another change, this time on the meta-discursive level, must take place: More often than not, the rules that govern interlocutors' discursive decisions will evolve as well. Such change must certainly happen in children's discourse on numbers if they are to be able to decide which of the two ways of multiplying positive by negative – the one offered by Sophie or the one designed by Roy should be accepted as the proper one.

Indeed, let us pause for a moment and ask ourselves what the children need in order to decide between the two possibilities. That the task is demanding is evidenced, among others, be the following testimony of the French writer Stendhal, who recalls the difficulties he experienced as a student when trying to find out the reasons for the related rule, "negative times negative is positive":

> "I thought that mathematics ruled out all hypocrisy… Imagine how I felt when I realized that no one could explain to me why *minus times minus yields plus….* That this difficulty was not explained to me was bad enough …. What was worse was that *it was explained to me by means of reasons that were obviously unclear to those who employed them."* (quoted in Hefendehl-Hebeker, 1991, p. 27).

It is noteworthy that Stendhal's complaint is about the nature of the justification he has heard rather than about the absence thereof. Let us try to figure out what this justification could be. Here is one possibility. Taking as a point of departure the request that the basic laws of numbers, as have been known so far, should not be violated, and assuming that the law "*plus times minus is minus*" and the rule $-(-x) = x$ have already been derived from these laws (Stendhal seemed to have had no problem with these ones!), the explainer may now argue that for any two positive numbers, a and b, the following must hold.

On the one hand,

(1) $$0 = 0 \cdot (-b) = [a + (-a)]\,(-b)$$

and on the other hand, because of the distributive law which is supposed to hold,

(2) $$[a + (-a)]\,(-b) = a(-b) + (-a)(-b)$$

Since it was already agreed that $a(-b) = -ab$, we get from (1) and (2):

$$-ab + (-a)(-b) = 0$$

From here, and from the law $-(-x) = x$, one now gets:

$$(-a)(-b) = -(-ab) = ab$$

One may say that it is the degree of formality of this argument that makes it unconvincing in the eyes of the student. Thus, let us try to imagine an alternative. A different, more effective kind of explanation could only come from everyday discourse. Indeed, secondary-school students' classroom conversations, not yet a case of a fully-fledged mathematical discourse, are typically a result of cross-breeding between everyday discourse and modern mathematical discourse. In the everyday discourse, claims about objects count as acceptable (*true*) if they seem necessary and inevitable, and if they are conceived as stating a property of a mind-independent 'external world'. This applies not only to material objects, but also to numbers, geometrical forms and all other mathematical entities to be implicated in colloquial uses. It is this "external reality" which is for us a touchstone of inevitability and certainty. In mathematics, like in everyday discourse, the student expects to be guided by something which can count as being beyond the discourse itself and existing independently of human decisions. This is what transpires from the words of the student by the name Dan who tried to explain to me his difficulty with the negatives:

Episode 5: Dan explains his difficulty with negative numbers

[1]	Dan :	*Minus is something that people invented*. I mean… we don't have anything in the environment to show it. I can't think about anything like that.
[2]	Anna:	Is everything that regards numbers invented by people?
[3]	Dan:	No, not everything…
[4]	Anna:	For instance?
[5]	Dan:	For example, the basic operation of addition, one plus one [is two] and *according to the logic of the world* this cannot be otherwise.
[6]	Anna:	And half plus one-third equals five sixths. Does it depend on us, humans or…
[7]	Dan:	Not on us. You can show it in the world.
[8]	Anna:	I see… and 5 minus 8 equals –3. It's us or not us?
[9]	Dan:	It's us.
[10]	Anna:	Why?
[11]	Dan:	Because in our world there is no example for such a thing.

Thus, the safest way for the student toward understanding and accepting the negative numbers and the operations on these numbers would be to make them a part of his or her everyday discourse. Alas, in the present case this does not seem possible. Although people usually can incorporate negative numbers into sentences that concern everyday matters, these discursive appearances are incomplete in that they rarely include operations on numbers and thus, in fact, refer to such entities as -2, or -10.5 as labels rather than fully-fledged numbers. This is evidenced by the results of my experiment in which eighteen students who have already learned about negative numbers were asked to construct sentences with the number -3, as well as questions the answers to which could be -2. In both cases, they were encouraged to look for utterances with “everyday content”. As can be seen from the results presented in Fig. 5, not all the students were up to the task. The few "everyday uses" of negative numbers were made solely in the context of temperature, latitude and bank overdraft. In all these cases, the negative numbers were applied as a label rather than as a measure of quantity.[8]

Sentence with –3: - The temperature went down to -3. (42%)
Questions the answer to which may be -2: - Temperature went down 12 degrees from 10 degrees. What is the temperature now? (42%) - How much money do you owe [*sic*!] to John? (25%)

Fig. 5: . Examples of “everyday” utterances involving negative numbers given by students

We are then left with the justification grounded in meta-discursive rule that underlied the formal argument presented above. According to this rule the only criterion for the extension of the existing mathematical discourse is the inner coherence of the resulting extended discourse. More specifically, all that is needed to accept a proposed use of a new type of number is its consistency with certain properties of the sets of numbers this definition is going to broaden. There is little chance that the young students for whom mathematical discourse is a description of the existing reality rather than the means for creating a new one will accept this new meta-rule, let alone re-invent it by themselves. (This, by the way, may well be the reason why teaching the negative numbers has been grounded for ages in the didactic principle epitomized in this unforgettable rhyme “Minus times minus is plus, the reason for this we need not discuss”; W.H. Auden, quoted in Kline 1980).

To see how true it is that the students may have no grounds for preferring one old template over another, let us go back to the seven graders whom we left puzzling over the question what should be the result of multiplying a positive by negative. The debate went on for two full periods and the class got eventually convinced by Roy who claimed that the sign should be like that of the multiplier with the bigger absolute value. The teacher seemed quite desperate.

Episode 6: Why choose one template rather than another?

[N46] Teacher:	You keep repeating what Roi said last Monday, and I want to know *why* you think it must be so.
[N47] Yoaz:	Because this is what Roi said.
[N48] Teacher:	But Roi himself didn’t explain why it is the magnitude that counts.
[N49] Roi:	Because there must be a law, one rule or another
[N50] Teacher:	There must be some rule, so it is better that we do it according to magnitude?
[N51] Leegal:	The bigger is the one that counts
………..	
[N83] Teacher:	Why [did you agree that six times is twelve]? Six times negative two is negative twelve – is this too complicated?
[N84] Roi:	But I am more charismatic… I managed to influence them all.

The picture we get from here is as follows: The children know that if they deal with numbers, there must be rules; and yet, they have no idea where these rules should come from. Their helplessness

[8] It is also noteworthy that many of the "everyday" questions to which the answer was supposed to be -2 suffered from out-of-focus syndrome; that is, although the negative quantity was somehow involved in the situation presented in the question, the actual answer to the question should be 2 rather than -2 (see the last example in Figure 5).

finds it expression in Roy's humorous declaration that the personality of the inventor of a definition is as good a reason for the acceptance of this definition as any other (after all, don't we repeat, time and again, that mathematics is a human creation?!). If you think about it, whatever change in meta-rules is to occur, it can only be initiated by the teacher. Indeed, unlike the object-level rules of mathematical discourse, the meta-level rules cannot be shown to be necessary or inevitable. As eloquently argued by Wittgenstein (1953; see also Sfard 2000b), in a certain deep sense they are but conventions. These conventions have their historical reasons, but the historical reasons are not today's students' reasons. The possibility that the students would re-construct these rules for themselves is thus highly implausible. Children can only arrive at these rules by interacting with an expert participant, at least part of the time.

This last didactic suggestion sounds quite straightforward. And yet, if we now go back to the first graders who are learning about triangles, we will see that children do not easily accept changes in meta-discursive rules even if the initiative comes from a very determined teacher.

Episode 7: Trying to convince Shira that shape C is a triangle

[T35] Teacher:	But you told me… Hold on, you told me that triangle… well…. Eynat, you told me, and Shira agreed, that that in triangle there must be three lines, right?
[T36] Eynat:	Right.
[T37] Teacher:	So, come on, tell me how many lines do we have here? [*in the shape presented in Fig. 2*]
[T38] Shira:	One, two, three….
[T39] Teacher:	So, maybe this *is* a triangle? So, Eynat erased [the circle] for no reason? You are not sure. About this one you said it is triangle [*shows another, more "canonic" triangle*].
[T40] Shira:	Because it is wide and it fits to be a triangle. It is not thin like a stick [*illustrates with hand movements and laughs*]
[T41] Teacher:	How do we know that a triangle… whether a shape is triangle? What did we say? What did we say? To say that shape fits to be triangle, what do we need?
[T42] Shira:	Three points… three vertices… and…
[T43] Teacher:	Three vertices and…?
[T44] Shira:	Three sides.
[T45] Teacher:	And three sides. Good. If so, this triangle [*!*] fits. Look, one side… and here I have one long side, and here I have another long side. So, here we have a triangle.
[T46] Shira:	And one vertex, and a second vertex, and a… point?!
[T47] Teacher:	Look here: one vertex, second vertex, third vertex
[T48] Shira:	So it *is* a triangle?

In this episode, the long debate on the status of the stick-like shape reaches its climax. To show the difficulty of the required discursive change let me analyze the brief conversation while trying to answer a number of questions, as specified below.

1. How do the meta-rules of the children's discourse have yet to change? A partial answer to this question has already been given in brief when I was analyzing *Episode 2*. Let me repeat and elaborate. The meta-rule that has to change is the one that regulates children's activity of giving names to geometrical shapes. At present, Eynat and Shira perform the naming task unreflectively, on the basis of their previous visual experiences. They recognize triangles and squares the way they recognize people's faces, that is in an intuitive way and without giving the reasons for their choices. From now on, they will be requested to communicate to others not only their decisions but also the way these decisions were made. This can only be done in words. The introduction of language imposes a linearity: Rather than satisfying themselves with the holistic visual impression which cannot be communicated to another person, the girls will have to tell their interlocutors how a shape should be scanned before the decision regarding its name is made. The scanning procedures (which I have once called an *attended focus*; see Sfard 2000c) are mediated by, and documented in, language. In fact, they are only possible as a part of verbal communication. When we check whether a shape is a triangle, we have to count its sides. The counting is a linguistic act and the result of counting is a word (*three*, in the case of triangles). The new way of making decisions about the names of geometric figures will thus be done *by analyzing words associated with the shapes* and without any reference to the overall visual impressions that were the basis of the naming procedures so far.

This new meta-discursive rule entails a change of yet another meta-discursive principle. So far, giving names has been an act of splitting the world into *disjoint* sets of objects. I highlighted the word *disjoint* in this last sentence to stress that within the initial geometric discourse, different names mean different objects. This fact is of crucial importance, as it entails the meta-rule according to which one cannot call a shape both *triangle* and *stick*, or both *square* and *rectangle*. This will have to change once the naming decisions are based on the results of detailed scanning procedures of varying complexity rather than on holistic visual appreciation. Indeed, scanning procedures can be ordered according to the relation of inclusion (for example, the procedure for identifying rectangles is, quite literally, a part of the procedure for identifying squares), and thus may be hierarchically organized. The hierarchical organization of the scanning procedures becomes, in turn, a basis for hierarchical categorization of geometrical shapes.

How does the teacher try to induce this change? The transition from the old to new meta-discursive rules must clearly take place before Eynat and Shira become fully convinced that stick-like shape is a triangle. Impatient to see the transition happening, the teacher repeatedly reminds the criterion which should be used in deciding: First she says that "in triangle there must be three lines" (T35), then she asks "To say that shape fits to be triangle, what do we need?" (T41), and finally eagerly confirms Shira's answer "three vertices and three sides" (T42). Although none of the teacher's formulations explicitly indicates the fact that having three sides is a sufficient condition for a shape to be a triangle, the sufficiency is there, signaled by the teacher with non-direct means. Over and over again the teacher initiates scanning the shapes and counting their elements. Invariably, the words "one, two, three" are followed with the telling "So.." (see T24, T39, T45), and, eventually, with the statement asserting that the shape is triangle. The word "so" is very effective in suggesting that whatever comes next is an inevitable entailment of the "one, two three" sequence.

How successful is the teachers' effort? On the face of it, the teacher's method of discursive alignment works: Shira soon learns to complete the procedure of counting to three with the words "So, this is a triangle" (children are incredibly dexterous at detecting and picking up discursive patterns!). In T48 she hurries to state this conclusion on her own accord, clearly aware of the rules of the game set by the teacher. And yet, the fact that she utters this conclusion as a question rather than as a firm assertion signals that she may be declaring a surrender rather than a true conviction. The girl knows *what* she is expected to say, but she does not know *why*. The claim that shape C is a triangle contradicts the meta-discursive rule according to which she has been making her naming decisions so far. What the teacher considers to be a necessary and sufficient condition for "triangleness", for the girl is a necessary condition, at best. The lack of certainty can be felt also in Eynat's contributions, in spite of her evident awareness of the meta-discursive rules that elude her partner (see T3 and T31). The ultimate evidence for the fact that old meta-discursive habits die hard will come some time later, when the children are asked to distinguish between rectangles and other polygons. Both girls will then adamantly reject the teacher's suggestion that a square can also be called rectangle and they will stick to their version for a long time in spite of the teacher's insistence.

One could conjecture that in the case we have been analyzing, the slowness of learning resulted not so much from the stubbornness of the old discursive habits as from the ineffectiveness of the teaching method. Moreover, since this method was based on demonstrating the application of the new meta-rules rather than on arguing for them explicitly, some people may criticize the teacher for violating the principle of learning with understanding. This is thus the proper place to remind ourselves that unlike the object-level rules of mathematics, each of which is logically connected to all the others, the meta-rules are not dictated by the logical necessity. In consequence, one cannot justify them in a truly convincing, rational way. The children, if they wish to communicate with others, will have to accept these rules just because they regulate the game played by more experienced players.[9] They will have to become participants of the new discourse before they can fully appreciate its advantages.

5 Final remarks: How does all this affect the practice?

In this talk I proposed to think about learning mathematics as developing a special type of discourse. It is now time to demonstrate that suggesting this conceptual shift was not a mere intellectual exercise. In this last section I wish to argue that the change of perspective is bound to affect some of our beliefs on teaching mathematics.

Let me begin with the nowadays popular principle according to which whatever learning we are trying to induce, we should keep it *meaningful* all along the way. The slogan *learning with understanding* stresses the importance and primacy of conceptual understanding over formalization and skill. As I mentioned several times, the conceptualization of learning as an introduction to a discourse leads me to question this insistence on the sustained understanding. From what has been said so far, I would rather conclude that one can make sense of mathematical discourse only through a persistent participation, and not prior to it. In fact, too great a stress on understanding may eventually become counterproductive, as it is likely to undermine students' willingness to engage in mathematical discourse at times of insufficient understanding. Let me thus reiterate the advice given by Cardan more than half a millenium ago to those who criticized him for his use of the "imaginary numbers": Whoever wishes to become fully fluent in mathematical communication has to persist in practicing mathematical discourse, and must do it "putting aside the mental tortures involved", if necessary.

Since, indeed, objectified mathematical discourse can only arise in the delicate dialectic between engaging in the mathematical communication and trying to understand, another popular pedagogical view must be questioned. Because of the existence of calculators and computers, some writers have been insisting that the student may be exempted from trying to attain procedural efficiency (see e.g. Devlin 1997**.** Some educators would go so far as to say that formalism and skills may be completely removed from school curricula. And yet, as I was arguing along these pages, it is a mistake to think of symbolization as a matter of finishing touches – of giving new "expression" to the "old thought". Rather, the special symbolic mediators are necessary to generate the mathematical communication in the first place.

According to yet another popular claim, the best way to assure effective learning is to keep mathematics embedded in real-life context. In discursive terms, this means that school mathematical discourse should always remain a part of everyday discourse. While discussing the learning of negative numbers I have just shown how unrealistic this goal is. Besides, if learning mathematics means an initiation to a special type of discourse, staying within the confines of the everyday discourse would contradict our aim!

Perhaps the most widely accepted assumption about learning is that the student is the builder of his or her own knowledge. This Piagetian claim is often misinterpreted as saying that children should construct their knowledge more or less on their own, in the course of collaborative problem solving. In discursive terms, this would mean that the students are expected to develop mathematical discourse while interacting with each other. Our data, followed by the discussion on the notion of meta-discursive rule and on the ways in which these rules evolve have shown the untenability of this belief as well.

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[9] This means that new meta-rules can only be dictated by the teacher. Let me immediately add that the kind of dictation we are talking about is not necessarily an imposition. As long as all the parties involved are willing to play the game, the introduction of its rules by an experienced player cannot be considered as violating anybody's freedom. More often than not, this is the case in the classroom situation: The teacher is willing to teach, whereas the children, even if they do not seem too eager, are willing to learn. Indeed, both sides are keen on having an effective communication. It is also tacitly agreed by all parties involved that these are the children who should adjust their discursive ways to those of the teacher, and not the other way round. After all, if the children were not ready to follow the discursive lead of the grownups, they would never become able to communicate with other people. So, it is only understandable that the children change their discursive ways by reading meta-discursive hints, by guessing what is proper and by imitating patterns appearing in the discursive actions of other interlocutors. All this is done for no other reason than the wish to improve communication by aligning themselves with their more experienced partners.

To sum up, the communicational vision of learning implies that while teaching mathematics, we should keep in mind the inherent difficulty of the endeavor, stemming from the endemic circularity of the learning process and from the fact that at least some of the meta-rules of mathematical discourse are not logically inevitable. Neither of the resulting dilemmas can be overcome by a purely rational effort. Coming to terms with negative numbers or with meta-discursive rules that change what seems to be a law of nature requires time and patience. This may be why the mathematician von Neumann has been heard saying to a journalist "One does not understand mathematics, young man, one just gets used to it". Naturally, this statement should not be taken as a serious, or sufficient, basis for a pedagogical advice. And yet, the communicational approach proposed in this talk did show the importance of discursive habits and the impossibility of developing them on a purely rational basis. Consequently, this approach brought with it a number of practical suggestions, some of which appear quite different from what is being practiced in schools these days.

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